Spectral sequences aren't scary

CARES

Eric (UARK)

https://locallyringed.space

24 October 2022

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Outline

- Core idea
- Five lemma
- Snake lemma
- Balancing Tor/Ext
- Composing derived functors

For simplicity, everything we write will just be modules over a fixed ring R.

Resources: Weibel, Vakil, McCleary.

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$$\begin{array}{ccccc} & \vdots & \vdots & \vdots & \\ \cdots & E^{-1,1} & E^{0,1} & E^{1,1} & \cdots \\ \cdots & E^{-1,0} & E^{0,0} & E^{1,0} & \cdots \\ \cdots & E^{-1,-1} & E^{0,-1} & E^{1,-1} & \cdots \\ & \vdots & \vdots & \vdots & \end{array}$$

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$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow{\scriptstyle d^{\uparrow}} & & \uparrow{\scriptstyle d^{\uparrow}} & & \uparrow{\scriptstyle d^{\uparrow}} & & \\
\cdots & \xrightarrow{d^{\rightarrow}} & E^{-1,1} & \xrightarrow{d^{\rightarrow}} & E^{0,1} & \xrightarrow{d^{\rightarrow}} & E^{1,1} & \xrightarrow{d^{\rightarrow}} & \cdots \\
 & & \uparrow{\scriptstyle d^{\uparrow}} & & \uparrow{\scriptstyle d^{\uparrow}} & & \uparrow{\scriptstyle d^{\uparrow}} & & \\
\cdots & \xrightarrow{d^{\rightarrow}} & E^{-1,0} & \xrightarrow{d^{\rightarrow}} & E^{0,0} & \xrightarrow{d^{\rightarrow}} & E^{1,0} & \xrightarrow{d^{\rightarrow}} & \cdots \\
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$$\begin{array}{ccc}
E^{p,q+1} & \xrightarrow{d^{\rightarrow}} & E^{p+1,q+1} \\
{\scriptstyle d^{\uparrow}}\big\uparrow & \text{anti} & \big\uparrow{\scriptstyle d^{\uparrow}} \\
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Why *anti*-commute?

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Given a double complex $\{E^{pq}\}$ we can construct a (single) complex called the **total complex** $\operatorname{Tot} E^{pq}$:

$$(\operatorname{Tot} E^{pq})^n := \bigoplus_{p+q=n} E^{pq}$$

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Notice that this is a complex, because

$$\begin{aligned}(d_{\mathrm{Tot}})^2 &= \left(d^{\uparrow} + d^{\rightarrow}\right)^2 \\ &= \left(d^{\uparrow}\right)^2 + \left(d^{\uparrow} d^{\rightarrow} + d^{\rightarrow} d^{\uparrow}\right) + (d^{\rightarrow})^2 \\ &= 0.\end{aligned}$$

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That's why!

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The tool is a **spectral sequence**.

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A **spectral sequence** (rightward) is a sequence of double complexes $({}_{\to}E_r^{pq})_{r\in\mathbf{Z}}$:

$$ {}_{\to}E_0^{pq}, {}_{\to}E_1^{pq}, {}_{\to}E_2^{pq}, \ldots $$

which we call **pages**.

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These ${}_{\to}d_r^{pq}$ maps identify ${}_{\to}E_{r+1}^{pq}$ with the cohomology of ${}_{\to}d_r^{pq}$ at ${}_{\to}E_r^{pq}$; that is,

$$ {}_{\to}E_{r+1}^{pq} \cong \frac{\ker {}_{\to}d_r^{pq}}{\operatorname{im} {}_{\to}d_r^{p+r-1,q-r}}. $$

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This pre-subscript "$\to$" nonsense is just notational to distinguish from a different filtration to come, and we'll drop it basically immediately. Sorry!

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This definition seems fiddly and technical, but it's easier to visualize. The differential on page r is $d_r^{pq} : E_r^{pq} \to E_r^{p-r+1,q+r}$. Here's the picture:

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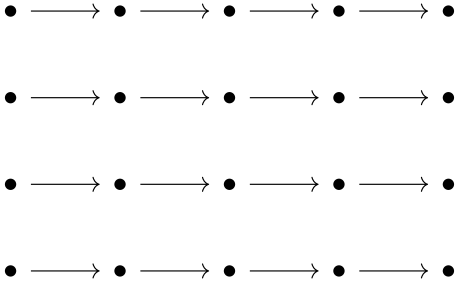
Page 0: $d_0 : E_0^{pq} \to E_0^{p+1,q}$



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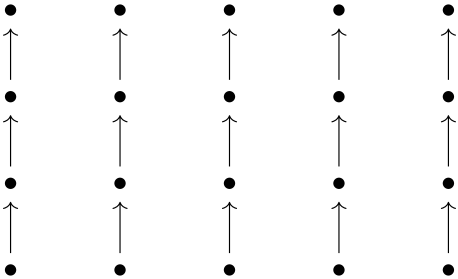
Page 1: $d_1 : E_1^{pq} \to E_1^{p,q+1}$



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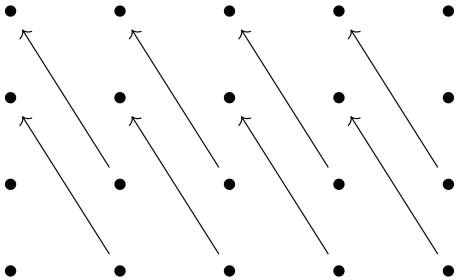
Page 2: $d_2 : E_2^{pq} \to E_2^{p-1,q+2}$



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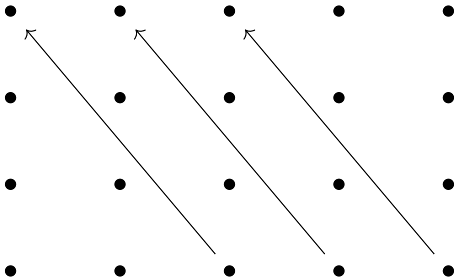
Page 3: $d_3 : E_3^{pq} \to E_3^{p-2,q+3}$



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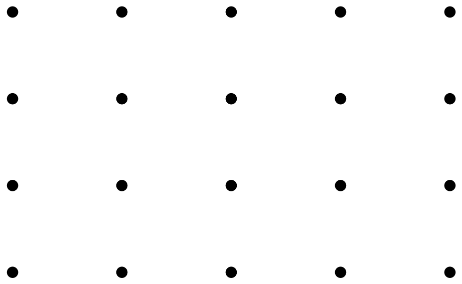

Et cetera.

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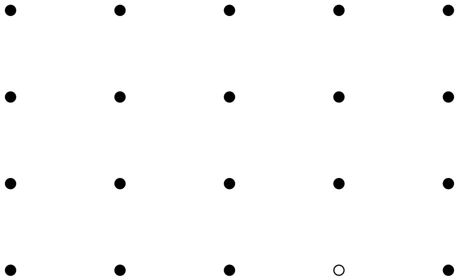

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$$\begin{array}{ccccccccc}
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \bullet \\
\\
\bullet & \longrightarrow & \bullet & \longrightarrow & \bullet & \longrightarrow & \circ & \longrightarrow & *
\end{array}$$

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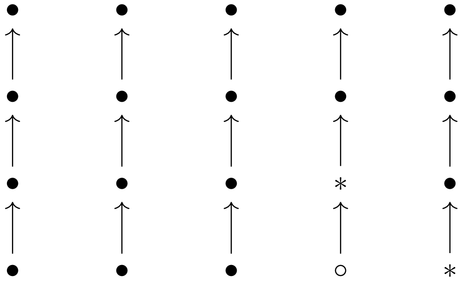

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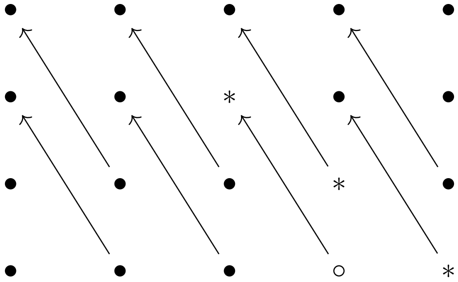

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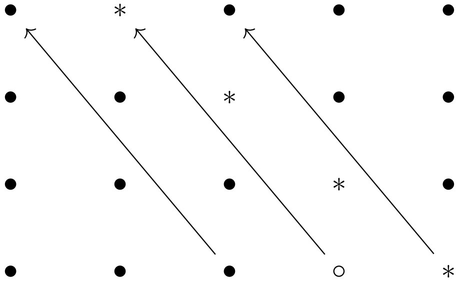

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$$
\begin{array}{ccccc}
\bullet & * & \bullet & \bullet & \bullet \\
\bullet & \bullet & * & \bullet & \bullet \\
\bullet & \bullet & \bullet & * & \bullet \\
\bullet & \bullet & \bullet & \circ & *
\end{array}
$$

We're ranging over all terms that are degree +1 from $\circ$.

Core idea

Recalling that $(\operatorname{Tot} E^{pq})^n = \oplus_{p+q=n} E^{pq}$, intuitively, we're taking the "$\circ$" term and mapping it to all the factors that d_{Tot} (also degree $+1$) does!

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By virtue of the fact that the cohomology of the differential d_r^{pq} defines the next page, E_{r+1}^{pq}, (a fact we glazed over when we just wrote a bunch of dots), intuitively, we're getting the cohomology of $\mathrm{Tot}\, E^{pq}$ by iteratively hacking away at the factors in the sum.

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(This is *certainly* just a loosey-goose vibes-only explanation at the moment, but it **can** be articulated in a precise way!)

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Theorem. *There is a filtration of the nth cohomology of* $\operatorname{Tot} E^{pq}$ *by* E_∞^{pq} *where* $p + q = n$.

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Theorem. *There is a filtration of the* n*th cohomology of* $\operatorname{Tot} E^{pq}$ *by* E_∞^{pq} *where* $p+q=n$.

But what do we mean by E_∞^{pq}? We only have E_r^{pq} for $r \in \mathbf{Z}$. The complete answer requires the notions of "convergence" which I am sweeping under the rug, but I will give the idea:

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0	$\bullet$	$\bullet$	$\bullet$	$\bullet$
0	$\bullet$	$\bullet$	$\bullet$	$\bullet$
0	$\bullet$	$\bullet$	$\bullet$	$\bullet$
0	0	0	0	0

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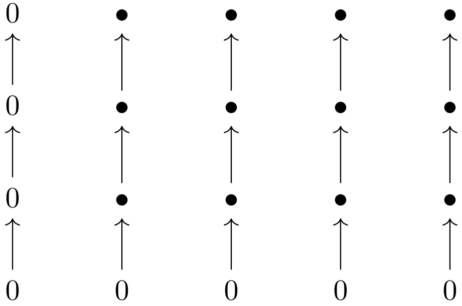

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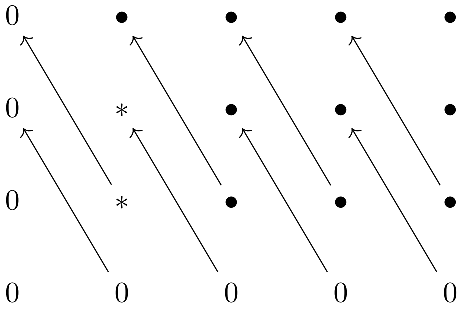

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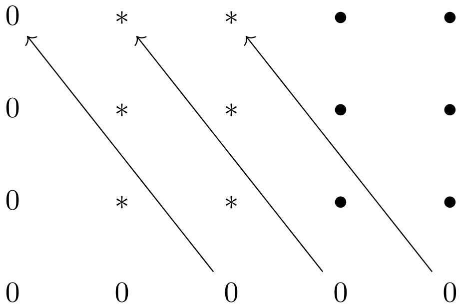

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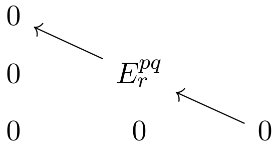
Since the entire double complex is first quadrant, given any (p, q), there is a page r large enough so that the differential on that page goes



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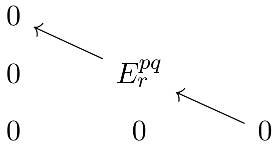

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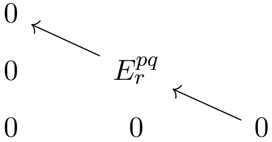

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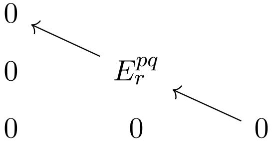

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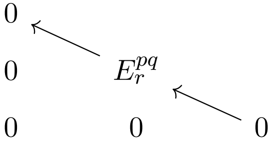

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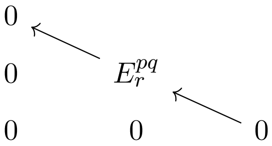

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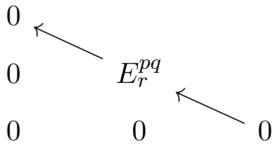

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Outside of the first quadrant setting, convergence might be more delicate, but we won't worry too much about such examples here.

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You should ask: what was the nonsense with the pre-subscript arrows ${}_{\rightarrow}E_r^{pq}$ and ${}_{\rightarrow}d_r^{pq}$ which we immediately forgot?

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It's to distinguish from the fact that everything we did was symmetric, so you just as easily could have the following definition:

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A **spectral sequence** (upward) is a sequence of double complexes $({}_{\uparrow}E_r^{pq})_{r\in\mathbf{Z}}$:

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$$ {}_{\uparrow}E_0^{pq}, {}_{\uparrow}E_1^{pq}, {}_{\uparrow}E_2^{pq}, \ldots $$

which we call **pages**. Each page r has a differential that goes

$$ {}_{\uparrow}d_r^{pq} : {}_{\uparrow}E_r^{pq} \to {}_{\uparrow}E_r^{p+r,q-r+1}. $$

These ${}_{\uparrow}d_r^{pq}$ maps identify ${}_{\uparrow}E_{r+1}^{pq}$ with the cohomology of ${}_{\uparrow}d_r^{pq}$ at ${}_{\uparrow}E_r^{pq}$; that is,

$$ {}_{\uparrow}E_{r+1}^{pq} \cong \frac{\ker {}_{\uparrow}d_r^{pq}}{\operatorname{im} {}_{\uparrow}d_r^{p-r,q+r-1}}. $$

Core idea

Note that the differential on page r is different! Now we have $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$. Here's the new picture:

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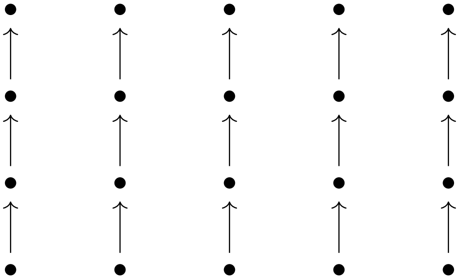
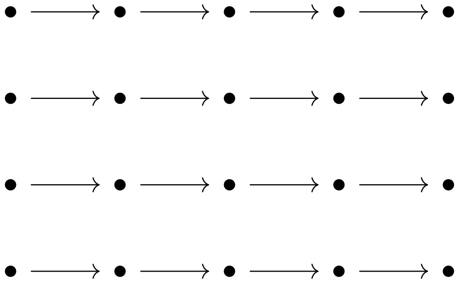
Page 0: $d_0 : E_0^{pq} \to E_0^{p,q+1}$



Core idea

Note that the differential on page r is different! Now we have $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$. Here's the new picture:

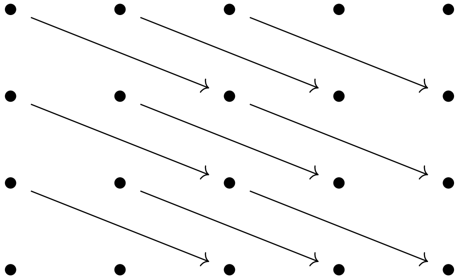
Page 1: $d_1 : E_1^{pq} \to E_1^{p+1,q}$



Core idea

Note that the differential on page r is different! Now we have $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$. Here's the new picture:

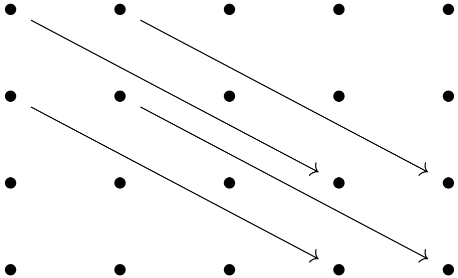
Page 2: $d_2 : E_2^{pq} \to E_2^{p+2,q-1}$



Core idea

Note that the differential on page r is different! Now we have $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$. Here's the new picture:

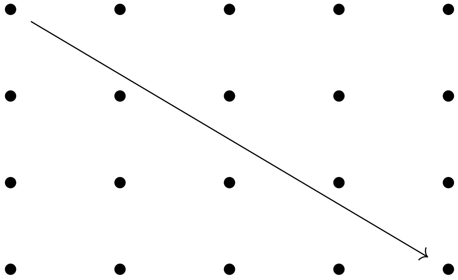
Page 3: $d_3 : E_3^{pq} \to E_3^{p+3,q-2}$



Core idea

Note that the differential on page r is different! Now we have $d_r^{pq} : E_r^{pq} \to E_r^{p+r,q-r+1}$. Here's the new picture:

Page 4: $d_4 : E_4^{pq} \to E_4^{p+4,q-3}$



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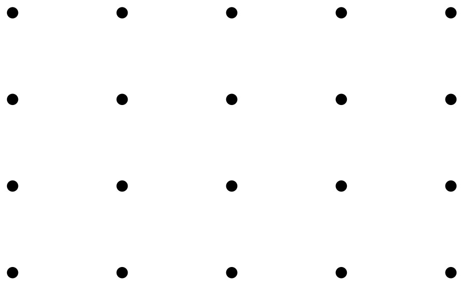

Et cetera.

Core idea

Here's the magic:

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This spectral sequence **also** produces a filtration of the nth cohomology of Tot E^{pq}! We still chip away at the degree n cohomology by mapping +1 into the direct sum. This has to be the case – it was just symmetry!

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We can do really cool calculations by fiddling with the right spectral sequence versus the up spectral sequence. When entire pages degenerate down to nothing (i.e., say ${}_{\rightarrow}E_{\infty}^{pq} = 0$), then the same must be true for ${}_{\uparrow}E_{\infty}^{pq}$!

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We can do really cool calculations by fiddling with the right spectral sequence versus the up spectral sequence. When entire pages degenerate down to nothing (i.e., say ${}_{\rightarrow}E_{\infty}^{pq} = 0$), then the same must be true for ${}_{\uparrow}E_{\infty}^{pq}$!

Let me show you, with examples:

Five lemma

Five lemma

Theorem. *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc} A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \\ \alpha\big\uparrow & & \beta\big\uparrow & & \gamma\big\uparrow & & \delta\big\uparrow & & \varepsilon\big\uparrow \\ A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \end{array}$$

if α, β, δ, and ε are isomorphisms, then so too is γ.

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if α, β, δ, *and* ε *are isomorphisms, then so too is* γ.

(In fact, the weaker version of the five lemma also follows from this argument. For the sake of simplicity in presentation, an exercise for the reader.)

Five lemma

Note: we **don't** actually care about the cohomology of the totalization of the double complex we just drew! Instead, we're just going to compare the two spectral sequences we have. They both converge to the cohomology of the total complex, but we don't care – we'll just use that they converge to the same thing.

Five lemma

Proof. Start with the right spectral sequence.

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Page 0:

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Five lemma

Proof. Start with the right spectral sequence.

Page 0:

$$A' \longrightarrow B' \longrightarrow C' \longrightarrow D' \longrightarrow E'$$

$$A \longrightarrow B \longrightarrow C \longrightarrow D \longrightarrow E$$

Because the rows are exact, page 1 is easy.

Proof. Start with the right spectral sequence.

Page 1:

$$\begin{array}{ccccc} Ker' & 0 & 0 & 0 & Cok' \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ Ker & 0 & 0 & 0 & Cok \end{array}$$

Because the rows are exact, page 1 is easy.

Five lemma

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A lot of terms just died, so page 2 is manageable.

Five lemma

Proof. Start with the right spectral sequence.

Page 2:

$$\begin{array}{ccccc} \widetilde{Ker'} & 0 & 0 & 0 & \widetilde{Cok'} \\ \\ \\ \widetilde{Ker} & 0 & 0 & 0 & \widetilde{Cok} \end{array}$$

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Every single map from page 2 onward is the zero map, so this is also page ∞.

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Page ∞:

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So far, anticlimactic, but let's see what happens with the up spectral sequence.

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Page 0:

$$\begin{array}{ccccc} A' & B' & C' & D' & E' \\ \alpha\big\uparrow & \beta\big\uparrow & \gamma\big\uparrow & \delta\big\uparrow & \varepsilon\big\uparrow \\ A & B & C & D & E \end{array}$$

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So far, anticlimactic, but let's see what happens with the up spectral sequence.

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A lot of these maps are isomorphisms, so page 1 will be nice.

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So far, anticlimactic, but let's see what happens with the up spectral sequence.

Page 1:

$$0 \longrightarrow 0 \longrightarrow \operatorname{coker}\gamma \longrightarrow 0 \longrightarrow 0$$

$$0 \longrightarrow 0 \longrightarrow \ker\gamma \longrightarrow 0 \longrightarrow 0$$

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Page ∞:

$$\begin{matrix} 0 & 0 & \operatorname{coker}\gamma & 0 & 0 \\ \\ 0 & 0 & \ker\gamma & 0 & 0 \end{matrix}$$

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Five lemma

But these two spectral sequences are supposed to converge to the same thing.

${}_{\rightarrow}E_{\infty}^{pq}$:

$\widetilde{K'}$	0	0	0	$\widetilde{C'}$
$\widetilde{K}$	0	0	0	$\widetilde{C}$

${}_{\uparrow}E_{\infty}^{pq}$:

0	0	$\operatorname{coker}\gamma$	0	0
0	0	$\ker\gamma$	0	0

Five lemma

But these two spectral sequences are supposed to converge to the same thing.

$_{\rightarrow}E_\infty^{pq}$:						$_{\uparrow}E_\infty^{pq}$:				
$\widetilde{K'}$	0	0	0	$\widetilde{C'}$		0	0	$\operatorname{coker}\gamma$	0	0
$\widetilde{K}$	0	0	0	$\widetilde{C}$		0	0	$\ker\gamma$	0	0

This forces $\operatorname{coker}\gamma = 0$ and $\ker\gamma = 0$, so γ is an isomorphism, as desired! □

Snake lemma

Snake lemma

Theorem. *Given a commutative diagram with exact rows*

$$\begin{array}{ccccccccc}
0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & 0 \\
 & & \alpha\Big\uparrow & & \beta\Big\uparrow & & \gamma\Big\uparrow & & \\
0 & \longrightarrow & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0
\end{array}$$

there is a long exact sequence

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma \longrightarrow \operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma \longrightarrow 0.$$

Snake lemma

Proof. The right spectral sequence:

Snake lemma

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Page 0:

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These rows are exact!

Snake lemma

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$$\begin{array}{ccccc} 0 & 0 & 0 & 0 & 0 \\ \uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\ 0 & 0 & 0 & 0 & 0 \end{array}$$

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This therefore must be page ∞.

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0	0	0	0	0
0	0	0	0	0

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Snake lemma

The up spectral sequence:

Snake lemma

The up spectral sequence:

Page 0:

$$\begin{array}{ccccc} 0 & A' & B' & C' & 0 \\ \uparrow & \alpha\uparrow & \beta\uparrow & \gamma\uparrow & \uparrow \\ 0 & A & B & C & 0 \end{array}$$

Snake lemma

The up spectral sequence:

Page 0:

$$\begin{array}{ccccc} 0 & A' & B' & C' & 0 \\ \uparrow & \alpha\uparrow & \beta\uparrow & \gamma\uparrow & \uparrow \\ 0 & A & B & C & 0 \end{array}$$

Take cohomology to get page 1.

Snake lemma

The up spectral sequence:

Page 1:

$$0 \longrightarrow \operatorname{coker}\alpha \xrightarrow{\varphi'} \operatorname{coker}\beta \xrightarrow{\psi'} \operatorname{coker}\gamma \longrightarrow 0$$

$$0 \longrightarrow \ker\alpha \xrightarrow{\varphi} \ker\beta \xrightarrow{\psi} \ker\gamma \longrightarrow 0$$

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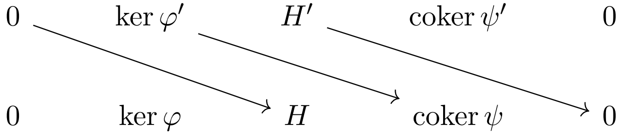
Take cohomology to get page 1.

Take cohomology to get page 2.

Snake lemma

The up spectral sequence:

Page 2:



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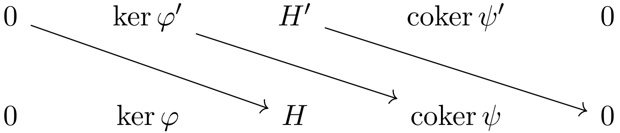
Snake lemma

The up spectral sequence:

Page 2:



Take cohomology to get page 1.

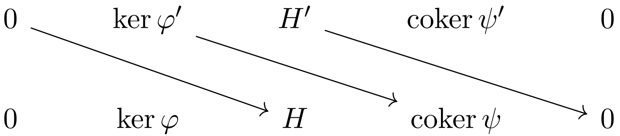
Take cohomology to get page 2.

Everything other than the map $\ker\varphi' \to \operatorname{coker}\psi$ stabilizes here. And then on page 3, everything stabilizes.

Snake lemma

The up spectral sequence:

Page 2:



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Everything other than the map $\ker\varphi' \to \operatorname{coker}\psi$ stabilizes here. And then on page 3, everything stabilizes.

Since ${}_{\to}E_{\infty}^{pq} = 0$, that means $\ker\varphi = H = H' = \operatorname{coker}\psi' = 0$, and that means $\ker\varphi' \to \operatorname{coker}\psi$ must be an isomorphism.

Snake lemma

If $\ker\varphi = H = H' = \operatorname{coker}\psi' = 0$, then, tracking back their definitions, this means

$$0 \longrightarrow \ker\alpha \longrightarrow \ker\beta \longrightarrow \ker\gamma$$

and

$$\operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma \longrightarrow 0$$

are exact.

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and

$$\operatorname{coker}\alpha \longrightarrow \operatorname{coker}\beta \longrightarrow \operatorname{coker}\gamma \longrightarrow 0$$

are exact.

If $\ker\varphi' \cong \operatorname{coker}\psi$, then

$$\ker\varphi' = \ker(\operatorname{coker}\alpha \to \operatorname{coker}\beta) \text{ and}$$
$$\operatorname{coker}\psi = \operatorname{coker}(\ker\beta \to \ker\gamma).$$

This defines the snake morphism $\ker\gamma \to \operatorname{coker}\alpha$.

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Balancing Tor/Ext

Balancing Tor/Ext

Theorem. *The left-derived functors*

$$\mathbf{L}_n(M \otimes_R -)(N)$$

and

$$\mathbf{L}_n(- \otimes_R N)(M)$$

are isomorphic; we call both $\mathrm{Tor}_n^R(M, N)$.

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Ext is similar (but careful of the contravariance!); consider it a fun exercise.

Balancing Tor/Ext

Proof. Choose a projective resolution of M:

$$\cdots \to P_3 \to P_2 \to P_1 \to P_0 \to M \to 0$$

Balancing Tor/Ext

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Choose a projective resolution of N:

$$\cdots \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to N \to 0$$

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Choose a projective resolution of N:

$$\cdots \to Q_3 \to Q_2 \to Q_1 \to Q_0 \to N \to 0$$

Computing a left-derived functor involves (independent of P or Q)

$$\mathbf{L}_n(- \otimes N)(M) := h_n(P \otimes N)$$
$$\mathbf{L}_n(M \otimes -)(N) := h_n(M \otimes Q)$$

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Choose a projective resolution of N:

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$$\mathbf{L}_n(- \otimes N)(M) := h_n(P \otimes N)$$
$$\mathbf{L}_n(M \otimes -)(N) := h_n(M \otimes Q)$$

We want to show these are isomorphic.

Balancing Tor/Ext

Given two complexes P and Q, you can build the tensor double complex:

$$\begin{array}{ccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & P_2 \otimes Q_0 & \longrightarrow & P_1 \otimes Q_0 & \longrightarrow & P_0 \otimes Q_0 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & P_2 \otimes Q_1 & \longrightarrow & P_1 \otimes Q_1 & \longrightarrow & P_0 \otimes Q_1 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
\cdots & \longrightarrow & P_2 \otimes Q_2 & \longrightarrow & P_1 \otimes Q_2 & \longrightarrow & P_0 \otimes Q_2 & \longrightarrow & 0 \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

Balancing Tor/Ext

Right spectral sequence:

Balancing Tor/Ext

Right spectral sequence:

Page 0:

$$\begin{array}{ccccccccc} & & 0 & & 0 & & 0 & & \\ \cdots & \longrightarrow & P_2 \otimes Q_0 & \longrightarrow & P_1 \otimes Q_0 & \longrightarrow & P_0 \otimes Q_0 & \longrightarrow & 0 \\ \cdots & \longrightarrow & P_2 \otimes Q_1 & \longrightarrow & P_1 \otimes Q_1 & \longrightarrow & P_0 \otimes Q_1 & \longrightarrow & 0 \\ \cdots & \longrightarrow & P_2 \otimes Q_2 & \longrightarrow & P_1 \otimes Q_2 & \longrightarrow & P_0 \otimes Q_2 & \longrightarrow & 0 \\ & & \vdots & & \vdots & & \vdots & & \end{array}$$

Balancing Tor/Ext

Right spectral sequence:

Page 1:

$$\begin{array}{ccccc}
 & 0 & 0 & 0 & \\
 & & & \uparrow & \\
\cdots & 0 & 0 & M \otimes Q_0 & 0 \\
 & & & \uparrow & \\
\cdots & 0 & 0 & M \otimes Q_1 & 0 \\
 & & & \uparrow & \\
\cdots & 0 & 0 & M \otimes Q_2 & 0 \\
 & & & \uparrow & \\
 & \vdots & \vdots & \vdots &
\end{array}$$

Balancing Tor/Ext

Right spectral sequence:

Page 2:

$$\begin{array}{cccccc} & 0 & 0 & 0 & \\ \cdots & 0 & 0 & h_0(M \otimes Q) & 0 \\ \cdots & 0 & 0 & h_1(M \otimes Q) & 0 \\ \cdots & 0 & 0 & h_2(M \otimes Q) & 0 \\ & \vdots & \vdots & \vdots & \end{array}$$

Balancing Tor/Ext

Right spectral sequence:

Page ∞:

$$
\begin{array}{ccccc}
 & 0 & 0 & 0 & \\
\cdots & 0 & 0 & h_0(M \otimes Q) & 0 \\
\cdots & 0 & 0 & h_1(M \otimes Q) & 0 \\
\cdots & 0 & 0 & h_2(M \otimes Q) & 0 \\
 & \vdots & \vdots & \vdots &
\end{array}
$$

Balancing Tor/Ext

Up spectral sequence:

Balancing Tor/Ext

Up spectral sequence:

Page 0:

$$
\begin{array}{ccccc}
 & 0 & 0 & 0 & \\
 & \uparrow & \uparrow & \uparrow & \\
\cdots & P_2 \otimes Q_0 & P_1 \otimes Q_0 & P_0 \otimes Q_0 & 0 \\
 & \uparrow & \uparrow & \uparrow & \\
\cdots & P_2 \otimes Q_1 & P_1 \otimes Q_1 & P_0 \otimes Q_1 & 0 \\
 & \uparrow & \uparrow & \uparrow & \\
\cdots & P_2 \otimes Q_2 & P_1 \otimes Q_2 & P_0 \otimes Q_2 & 0 \\
 & \uparrow & \uparrow & \uparrow & \\
 & \vdots & \vdots & \vdots &
\end{array}
$$

Balancing Tor/Ext

Up spectral sequence:

Page 1:

$$
\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
\cdots & \longrightarrow & P_2 \otimes N & \longrightarrow & P_1 \otimes N & \longrightarrow & P_0 \otimes N & \longrightarrow & 0 \\
\cdots & & 0 & & 0 & & 0 & & 0 \\
\cdots & & 0 & & 0 & & 0 & & 0 \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}
$$

Balancing Tor/Ext

Up spectral sequence:

Page 2:

$$\begin{array}{ccccc}
 & 0 & 0 & 0 & \\
\cdots & h_2(P \otimes N) & h_1(P \otimes N) & h_0(P \otimes N) & 0 \\
\cdots & 0 & 0 & 0 & 0 \\
\cdots & 0 & 0 & 0 & 0 \\
 & \vdots & \vdots & \vdots &
\end{array}$$

Balancing Tor/Ext

Up spectral sequence:

Page ∞:

$$\begin{array}{ccccc} & 0 & 0 & 0 & \\ \cdots & h_2(P \otimes N) & h_1(P \otimes N) & h_0(P \otimes N) & 0 \\ \cdots & 0 & 0 & 0 & 0 \\ \cdots & 0 & 0 & 0 & 0 \\ & \vdots & \vdots & \vdots & \end{array}$$

Balancing Tor/Ext

For ${}_{\to}E_2^{pq} = {}_{\to}E_\infty^{pq}$, the only term in total degree $p + q = -n$ is $h_n(M \otimes Q)$. Thus the filtration of cohomology by page ∞ is trivial and we get

$$h^{-n}(\mathrm{Tot}\, E^{pq}) \cong h_n(M \otimes Q) = \mathbf{L}_n(M \otimes -)(N).$$

Balancing Tor/Ext

For ${}_{\rightarrow}E_2^{pq} = {}_{\rightarrow}E_\infty^{pq}$, the only term in total degree $p+q=-n$ is $h_n(M \otimes Q)$. Thus the filtration of cohomology by page ∞ is trivial and we get

$$h^{-n}(\operatorname{Tot} E^{pq}) \cong h_n(M \otimes Q) = \mathbf{L}_n(M \otimes -)(N).$$

But similarly, for ${}_{\uparrow}E_2^{pq} = {}_{\uparrow}E_\infty^{pq}$, the only term in total degree $p+q=-n$ is $h_n(P \otimes N)$. Thus

$$h^{-n}(\operatorname{Tot} E^{pq}) \cong h_n(P \otimes N) = \mathbf{L}_n(- \otimes N)(M).$$

Balancing Tor/Ext

For ${}_{\rightarrow}E_2^{pq} = {}_{\rightarrow}E_\infty^{pq}$, the only term in total degree $p + q = -n$ is $h_n(M \otimes Q)$. Thus the filtration of cohomology by page ∞ is trivial and we get

$$h^{-n}(\operatorname{Tot} E^{pq}) \cong h_n(M \otimes Q) = \mathbf{L}_n(M \otimes -)(N).$$

But similarly, for ${}_{\uparrow}E_2^{pq} = {}_{\uparrow}E_\infty^{pq}$, the only term in total degree $p + q = -n$ is $h_n(P \otimes N)$. Thus

$$h^{-n}(\operatorname{Tot} E^{pq}) \cong h_n(P \otimes N) = \mathbf{L}_n(- \otimes N)(M).$$

By transitivity, $\mathbf{L}_n(M \otimes -)(N) \cong \mathbf{L}_n(- \otimes N)(M)$, as desired. $\square$

Composing derived functors

Composing derived functors

Theorem. [Grothendieck] *If F and G are left-exact functors and F sends injective objects to G-acyclic objects, then there is a spectral sequence whose page 2 is*

$$\rightarrow E_2^{pq} = \mathbf{R}^q G(\mathbf{R}^p F(M))$$

which converges to $\mathbf{R}^{p+q}(G \circ F)(M)$.

Composing derived functors

Proof. Choose an injective resolution of M:

$$0 \to M \to I^0 \to I^1 \to I^2 \to I^3 \to \cdots .$$

Composing derived functors

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Apply F termwise to I:

$$0 \to F(I^0) \to F(I^1) \to F(I^2) \to F(I^3) \to \cdots .$$

Composing derived functors

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Now, take a Cartan-Eilenberg resolution J of the complex $F(I)$,

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \vdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & J^{0*} & \longrightarrow & J^{1*} & \longrightarrow & J^{2*} & \longrightarrow & J^{3*} & \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ 0 & \longrightarrow & F(I^0) & \longrightarrow & F(I^1) & \longrightarrow & F(I^2) & \longrightarrow & F(I^3) & \longrightarrow \cdots \\ & & \uparrow & & \uparrow & & \uparrow & & \uparrow \\ & & 0 & & 0 & & 0 & & 0 \end{array}$$

then apply G termwise to J, to get the following double complex:

Composing derived functors

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G(J^{02}) & \longrightarrow & G(J^{12}) & \longrightarrow & G(J^{22}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G(J^{01}) & \longrightarrow & G(J^{11}) & \longrightarrow & G(J^{21}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & G(J^{00}) & \longrightarrow & G(J^{10}) & \longrightarrow & G(J^{20}) & \longrightarrow & \cdots \\
 & & \uparrow & & \uparrow & & \uparrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Composing derived functors

Let's calculate ${}_{\uparrow}E^{pq}$.

Composing derived functors

Let's calculate ${}_{\uparrow}E^{pq}$.

Page 0:

$$
\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
 & \uparrow & \uparrow & \uparrow & \\
0 & G(J^{02}) & G(J^{12}) & G(J^{22}) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
0 & G(J^{01}) & G(J^{11}) & G(J^{21}) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
0 & G(J^{00}) & G(J^{10}) & G(J^{20}) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
 & 0 & 0 & 0 &
\end{array}
$$

Composing derived functors

Let's calculate ${}_{\uparrow}E^{pq}$.

Page 1:

$$
\begin{array}{ccccccccc}
 & & \vdots & & \vdots & & \vdots & & \\
0 & & 0 & & 0 & & 0 & & \cdots \\
0 & & 0 & & 0 & & 0 & & \cdots \\
0 & \longrightarrow & GF(I^0) & \longrightarrow & GF(I^1) & \longrightarrow & GF(I^2) & \longrightarrow & \cdots \\
 & & 0 & & 0 & & 0 & &
\end{array}
$$

Composing derived functors

Let's calculate ${}_{\uparrow}E^{pq}$.

Page 2:

$$
\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
0 & 0 & 0 & 0 & \cdots \\
0 & 0 & 0 & 0 & \cdots \\
0 & h^0(GF(I)) & h^1(GF(I)) & h^2(GF(I)) & \cdots \\
 & 0 & 0 & 0 &
\end{array}
$$

Composing derived functors

Let's calculate ${}_{\uparrow}E^{pq}$.

Page ∞: Note that $h^n(GF(I))$ is, by definition, $\mathbf{R}^n(G \circ F)(M)$.

$$\begin{array}{ccccc} & \vdots & \vdots & \vdots & \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots \\ 0 & h^0(GF(I)) & h^1(GF(I)) & h^2(GF(I)) & \cdots \\ & 0 & 0 & 0 & \end{array}$$

Composing derived functors

Thus, via the upward spectral sequence, we get

$$ {}_{\uparrow}E_2^{pq} = {}_{\uparrow}E_\infty^{pq} = \mathbf{R}^{p+q}(G \circ F)(M). $$

We claimed we could compare this to $\mathbf{R}^q G(\mathbf{R}^p F(M))$. We'll see this composition fall out of the right spectral sequence.

Composing derived functors

Let's calculate ${}_{\rightarrow}E^{pq}$.

Composing derived functors

Let's calculate ${}_{\rightarrow}E^{pq}$.

Page 0:

$$\begin{array}{ccccccccc} & & \vdots & & \vdots & & \vdots & & \\ 0 & \longrightarrow & G(J^{02}) & \longrightarrow & G(J^{12}) & \longrightarrow & G(J^{22}) & \longrightarrow & \cdots \\ 0 & \longrightarrow & G(J^{01}) & \longrightarrow & G(J^{11}) & \longrightarrow & G(J^{21}) & \longrightarrow & \cdots \\ 0 & \longrightarrow & G(J^{00}) & \longrightarrow & G(J^{10}) & \longrightarrow & G(J^{20}) & \longrightarrow & \cdots \\ & & 0 & & 0 & & 0 & & \end{array}$$

Composing derived functors

Let's calculate ${}_{\rightarrow}E^{pq}$.

Page 1:

$$\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
 & \uparrow & \uparrow & \uparrow & \\
0 & h^0(G(I^{*2})) & h^1(G(I^{*2})) & h^2(G(I^{*2})) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
0 & h^0(G(I^{*1})) & h^1(G(I^{*1})) & h^2(G(I^{*1})) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
0 & h^0(G(I^{*0})) & h^1(G(I^{*0})) & h^2(G(I^{*0})) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
 & 0 & 0 & 0 &
\end{array}$$

Composing derived functors

Let's calculate ${}_{\rightarrow}E^{pq}$.

Page 1:

$$
\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
 & \uparrow & \uparrow & \uparrow & \\
0 & G(h^0(I^{*2})) & G(h^1(I^{*2})) & G(h^2(I^{*2})) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
0 & G(h^0(I^{*1})) & G(h^1(I^{*1})) & G(h^2(I^{*1})) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
0 & G(h^0(I^{*0})) & G(h^1(I^{*0})) & G(h^2(I^{*0})) & \cdots \\
 & \uparrow & \uparrow & \uparrow & \\
 & 0 & 0 & 0 &
\end{array}
$$

Composing derived functors

Let's calculate ${}_{\rightarrow}E^{pq}$.

Page 2: (who cares about maps now!)

$$\begin{array}{ccccc}
 & \vdots & \vdots & \vdots & \\
0 & h^2Gh^0F(I) & h^2Gh^1F(I) & h^2Gh^2F(I) & \cdots \\
0 & h^1Gh^0F(I) & h^1Gh^1F(I) & h^1Gh^2F(I) & \cdots \\
0 & h^0Gh^0F(I) & h^0Gh^1F(I) & h^0Gh^2F(I) & \cdots \\
 & 0 & 0 & 0 &
\end{array}$$

Composing derived functors

We don't need to know about the maps anymore, because we've just learned that

$$_{\rightarrow}E_2^{pq} = h^q G h^p F(I)$$

Composing derived functors

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$$
\begin{aligned}
{}_{\rightarrow}E_2^{pq} &= h^q G h^p F(I) \\
&= \mathbf{R}^q G(\mathbf{R}^p F(M))
\end{aligned}
$$

Composing derived functors

We don't need to know about the maps anymore, because we've just learned that

$$\begin{aligned} {}_{\rightarrow}E_2^{pq} &= h^q G h^p F(I) \\ &= \mathbf{R}^q G(\mathbf{R}^p F(M)) \end{aligned}$$

Since ${}_{\uparrow}E^{pq}$ and ${}_{\rightarrow}E^{pq}$ converge to the same thing, and we already learned that ${}_{\uparrow}E_{\infty}^{pq} = \mathbf{R}^{p+q}(G \circ F)(M)$, we now know that

$${}_{\rightarrow}E_2^{pq} = \mathbf{R}^q G(\mathbf{R}^p F(M))$$

converges to $\mathbf{R}^{p+q}(G \circ F)(M)$, as desired! $\square$